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A posteriori corrections for cubic and quintic interpolating splines at equally-spaced knots.

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ABSTRACT

The method proposed recently by Lucas [13], for the a posteriori correction of odd degree interpolating periodic splines is extended to non-periodic cubic and quintic splines.

1. Introduction

Let C_p^n [a,b] denote the class of all functions in C^n $(-\infty,\infty)$ which are periodic of period b-a. Also, let S denote the unique periodic spline of odd degree 2r-1 which interpolates $y \in C_p^n$ [a,b], $n \geq 0$ at the equally spaced knots

$$x_i = a + ih \ ;. \quad i = 0(1)k \ , \quad h = (b-a)/k \tag{1.1}$$

That is, S satisfies the following:

(i) In each interval (x_{i-1}, x_i); $i = 1(1)k$, S is a polynomial of degree at most 2r-1.

(ii) $S \in C^{2r-2}[a,b]$, and S satisfies periodic end conditions at a and b so that $\bar{S} \in C_p^{2r-2}$ [a,b], where $\bar{S}$ is the periodic extension of s.

(iii) $S(x_i) - y(x_i)$; $i = 0(1)k$. (1.2)

This paper is concerned with a method proposed recently by Lucas [13], for obtaining a posteriori improvements to such periodic spline interpolants. The method of [13] involves the use of polynomial type correction terms which, for sufficiently smooth y, increase the order of convergence of the spline approximations by several powers of h, For example, it is shown in [13] that if $y \in C_p^{2r+3}$ [a,b] then, for $0 \leq u \leq 1$,

$$\begin{aligned} y^{(j)}(x_i + \mu h) &= S^{(j)}(x_i + uh) \\ &+ \frac{h^{2r-j-2}}{(2r)!}\{\delta^2 S_i^{(2r-2)} p_0^{(j)}(\mu) \\ &+ \frac{(\delta^2 S_{i+1}^{(2r-2)} - \delta^2 S_{i-1}^{(2r-2)}}{2(2r+1)} P_1^{(j)}(\mu) \\ &+ \frac{\delta^4 S_i^{(2r-2)}}{2(2r+1)(2r+2)} P_2^{(j)}(\mu)\} + 0(h^{2r+3-j}) \ ; \\ &\qquad j = 0 \ (1) \ 2r, \end{aligned} \tag{1.3}$$

where the polynomials P_0, P_1 and P_2 are known in closed form in terms of

Bernoullli numbers. The estimate (1 .3) should be compared with the well-known result

$$y^{(j)}(x) = S^{(j)}(x) + 0(h^{2r-j}); \quad j = 0(1)2r-1 \quad , \tag{1.4}$$

which holds for $y \in C_p^{2r}[a,b]$.

In the present paper we consider the problem of extending the results of [13] to non-periodic cubic $(r = 2)$ and quintic $(r = 3)$ interpolating splines. In particular, we determine the precise criteria that the two (four) end conditions of a cubic (quintic) spline S must satisfy for (1.3) to hold in the interval [a+2h,b-h], and explain how the result can be extended over the whole of [a,b]. We also consider the effect that inferior end conditions have on the quality of the corrected approximations.

2. A posteriori corrections for cubic and quintic splines

Throughout this section, S will denote a non-periodic spline of degree 2r-1 , where $r = 2$ or $r = 3$, interpolating a function y at the equally spaced knots

$$x_i = a + ih; \quad i = 0(1)k, \quad h = (b-a)/k \quad . \tag{2.1}$$

That is, S will be either a cubic $(r = 2)$ or quintic $(r = 3)$ spline on [a,b], satisfying the k+1 interpolation conditions,

$$S(x_i) = y(x_i) \quad ; \quad i = 0(1)k \quad , \tag{2.2}$$

and an appropriate set of 2r-2; $r = 2,3$, end conditions. The following notation will also be used: $y_i^{(j)}$ and $S_i^{(j)}$ will denote $y^{(j)}(x_i)$ and $S^{(j)}(x_i)$, δ will denote the central difference operator, and $[\cdot]$ will Denote integer part . Finally, $\sum_{m=1}^{0}(\cdot)$ will be taken to mean that there are no terms under the summation sign.

The procedure used in this section, for extending the results of Lucas [13] to non—periodic cubic and quintic splines S of the type described above, emerges easily from the analysis contained in [13] In particular, the analysis of [13] indicates that we must first establish a number of results concerning the asymptotic expansions that the derivatives of S satisfy at the knots (2.1). These preliminary results are derived below, by making use of results already available in the literature; see [11], [12] and also [1], [5], [6] and [8],

We consider first the case where S is a cubic spline, and assume that $y \in C^8[a,b]$. Then, the well-known relations

$$S_{i-1}^{(2)} + 4S_i^{(2)} + S_{i+1}^{(2)} = \frac{6}{h^2}\delta^2 y_i \; ; \; i = 1(1)k-1 \; , \tag{2.3}$$

lead to

$$(S_{i-1}^{(2)} + y_{i-1}^{(2)}) + 4(S_i^{(2)} - y_i^{(2)}) + (S_{i+1}^{(2)} - y_{i+1}^{(2)}) = E_i \; ; \; i = 1(1)k-1 \; , \tag{2.4a}$$

where, by Taylor series expansion about x_i ,

$$\begin{aligned} E_i &= \frac{6}{h^2}\delta^2 y_i - (y_{i-1}^{(2)} + 4y_i^{(2)} + (y_{i+1}^{(2)}) \\ &= -\frac{h^2}{2} y_i^{(4)} - \frac{h^4}{15} y_i^{(6)} + 0(h^6) \end{aligned} \tag{2.4b}$$

This indicates that, under appropriate end conditions, $S_i^{(2)}$ will have an asymptotic expans ion of the form

$$(S_i^{(2)} + y_i^{(2)} + \alpha h^2 y_i^{(4)} + \beta h^4 y_i^{(6)} + 0(h^6)$$

In fact, if

$$\lambda_{3,1} = S_i^{(2)} - y_i^{(2)} - \alpha h^2 y_i^{(4)} - \beta h^4 y_i^{(6)}$$

then (2.4) gives

$$\lambda_{3,\, i-1} + 4\lambda_{3,i} + \lambda_{3,i+1} = \hat{E}_i \quad ;i = 1(1)k-1 \, ,$$

where

$$\hat{E}_i = -(6\alpha + \frac{1}{2})\, h^2 y_i^{(4)} - (6\beta + \alpha + \frac{1}{15})\, h^4 y_i^{(6)} + 0(h^6)$$

Therefore, if a = -1/12 and $\beta =$ 1/360, i.e. if

$$\lambda_{3,i} = S_i^{(2)} - y_i^{(2)} + \sum_{m=1}^{2} \ell_{3,m} h^{2m} y_i^{(2+2m)}, \qquad (2.5a)$$

where

$$\ell_{3,1} = 1/12 \quad \text{and} \quad \ell_{3,2} = -1/360 , \qquad (2.5b)$$

then

$$\lambda_{3,i-1} + 4\lambda_{3,i} + \lambda_{3,i+1} = 0(h^6) \;;\quad i = 1(1)k\text{-}1 \qquad (2.6)$$

This show that, under approprxate end conditions, $S_i^{(2)}$ will have the asymptotic expansion

$$S_i^{(2)} = y_i^{(2)} - \sum_{m=1}^{2} \ell_{3,m} h^{2m} y_i^{(2+2m)} + 0(h^6) . \qquad (2.7)$$

Let now S be a quintic spline, assume that $y \in C^{12}$ [a,b] and consider the problem of determining the asymptotic expansion of S_x. In this case by applying similar arguments to the above to the quintic spline relations

$$S_{i-2}^{(4)} + 26S_{i-1}^{(4)} + 66S_i^{(4)} + 26S_{i+1}^{(4)} + S_{i+2}^{(4)} = \frac{120}{h^4}\delta^4 y_i \;;\; i = 2(1)\; k-2 , \qquad (2.8)$$

we find that

$$\lambda_{5,1\text{-}2} + 26\lambda_{5,i-1} + 66\lambda_{5,i} + 26\lambda_{5,i+1} + \lambda_{5,i+2} = 0(h^8);\;\; i = 2(1)\; k\text{-}1 \qquad (2.9)$$

where

$$\lambda_{5,i} = S_i^{(4)} - y_i^{(4)} + \sum_{m=1}^{3} \ell_{5,m} h^{2m} y_i^{(4+2m)} , \qquad (2.10a)$$

with

$$\ell_{5,1} = 1/12 \;,\; \ell_{5,2} = -1/240 \;,\; \ell_{5,3} = 1/7{,}560 \qquad (2.10b)$$

That is, under appropriate end conditions, $S_i^{(4)}$. will have the asymptotic expansion

$$S_i^{(4)} = y_i^{(4)} - \sum_{m=1}^{3} \ell_{5,m} h^{2m} y_i^{(4+2m)} + 0(h^6) . \tag{2.11}$$

By allowing y to be of intermediate continuity, we can state the results (2.6) and (2.9) in more precise form as follows.

Lemma 2.1

(ii) *Cubic Spline S.* Let $y \in c^{4+n}[a,b]$, $n \geq 0$, let $N = \min\{[(n+1)/2],2\}$ and let

$$\lambda_{3,i}^{\{N\}} = S_i^{(2)} - y_i^{(2)} + \sum_{m=1}^{N} \ell_{3,m} h^{2m} y_i^{(2+2m)} \; ; \; i = 0(1)k , \tag{2.12a}$$

where

$$\ell_{3,1} = 1/12 \quad \text{and} \quad \ell_{3,2} = -1/360 , \tag{2,12b}$$

Then

$$\lambda_{3,i-1}^{\{N\}} = 4\lambda_{3,i}^{(N)} + \lambda_{3,i+1}^{(N)} = 0(h^{2+\hat{n}}) \; ; \; i = 1(1)k-1 \tag{2.13a}$$

where

$$\hat{n} = \min\{n,4\} . \tag{2.13b}$$

(ii) *Quintic spline S.* Let $y \in C^{6+n}[a,b]$, $n \geq 0$, let $N = \min\{[(n+1)/2],3\}$ and let

$$\lambda_{5,1}^{\{N\}} = S_i^{(4)} - y_i^{(4)} + \sum_{m=1}^{N} \ell_{5,m} h^{2m} y_i^{(4+2m)} \; ; \; i = 0(1)k , \tag{2.14b}$$

where

$$\ell_{5,1} = 1/12 , \; \ell_{5,2} = -1/240 \; \text{and} \; \ell_{5,3} = 1/7,560 \tag{2.14b}$$

Then,

$$\lambda_{5,i-2}^{\{N)} + 26\lambda_{5,i-1}^{(N)} + 66\lambda_{5,i}^{(N)} + 26\lambda_{5,i+1}^{(N)} + \lambda_{5,i+2}^{\{N)} = 0(h2 + \hat{n}) \;;$$

$$i = 2(1)k\text{-}2 \quad , \qquad (2.15a)$$

where

$$\hat{n} = \min\{n,6\} \quad . \square \qquad (2.15b)$$

Let $y \in C^{2r+n}[a,b]$. Then, the lemma shows that it is always possible to choose the end conditions of a cubic $(r=2)$ or quintic $(r=3)$ spline so that, with $N = \min\{[(n+1)/2],r\}$,

$$\max_{0 \le 1 \le k} \left\{ \left| \lambda_{2r-1,i}^{\{N\}} \right| \right\} = 0(h^{2+\hat{n}}) \;;\; \hat{n} = \min\{n,2r\} \quad .$$

For example, under the assumption $y \in c^{4r}[a,b]$, the best possible orders $0(h^{2r+1})$; $r = 2,3$, can be achieved by taking the cubic and quintic spline end conditions to be respectively

$$S_i^{(2)} = y_i^{(2)} - \sum_{m=1}^{2} \ell_{3,m} h^{2m} y_i^{(2+2m)} \;;\; i = 0,\, k \;, \qquad (2.16)$$

and

$$S_i^{(4)} = y_i^{(4)} - \sum_{m=1}^{4} \ell_{5,m} h^{2m} y_i^{(4+2m)} \;;\; 0,1,k-1,k \qquad (2.17)$$

The above observation motivates the following definition.

<u>Definition 2. 1</u> Let S be either a cubic $(r=2)$ or quintic $(r=3)$ spline. Then, the 2r-2 end conditions of S are said to be of order p if, for $y \in C^{4r}[a,b]$, the parameters $\lambda_{2r-1,i}^{\{r\}}$ defined by (2.12) and (2.14) satisfy

$$\max_{0 \le i \le k} \left\{ \left| \lambda_{2r-1,i}^{\{r\}} \right| \right\} = 0(h^{p}) \;,\; p > 0 \,. \square \qquad (2.18)$$

It follows at once from Lemma 2.1 that $p \le 2 + 2r$. In other words the "best" possible orders for cubic and quintic spline end conditions are respectively p=6 and $p=8$. It also follows that if $y \in C^{2r+n}[a,b]$ and the end conditions of S are of order $p \geq 2+\hat{n}$, where $\hat{n} = \min\{n,2r\}$, then

$$y_i^{(2r-2)} - S_i^{(2r-2)} = \sum_{m=1}^{N} \ell_{2r-1,m} h^{2m} y_i^{(2r-2+2m)} = 0(h^{2+\hat{n}})\ ;\ i=0(1)k, \qquad (2,19)$$

where, as in Lemma 2.1, $N=\min\{[(n+1)/2],r\}$ and the coefficients $\ell_{2r-1,m}$; $r=2,3$, are given respectively by (2.12b) and (2.14b).

<u>Lemma 2.2</u>

(i) *Cubic spline S.* Let $y \in C^{4+n}[a,b]$, $n \geq 0$, and let the end conditions of S be of order $p \geq 2+\hat{n}$, where $\hat{n} = \min\{n,4\}$. Then, with $N=\min\{[n/2],2\}$,

$$y_i^{(1)} - S_i^{(1)} = \sum_{m=1}^{N} \alpha_{3,m} h^{2+2m} y_i^{(3+2m)} + 0(h^{3+\hat{n}})\ ;\ i=0(1)k, \qquad (2.20a)$$

where

$$\alpha_{2,1} = 1/180 \quad \text{and} \quad \alpha_{3,2} = -1/1{,}512 \ . \qquad (2.20b)$$

(ii) *Quintio spline S.* Let $y \in C^{6+n}[a,b]$, $n \geq 0$, and let the end conditons of S be of order $p \geq 2+\hat{n}$, where $\hat{n} = \min\{n,6\}$. Also, let $N_1 = \min\{[n/2],3\}$ and $N_2 = \min\{[(n+1)/2],3\}$. Then:

$$\text{(a)} \quad y_i^{(1)} - S_i^{(1)} = \sum_{m=1}^{N} \alpha_{5,m} h^{4+2m} y_i^{(5+2m)} + 0(h^{5+\hat{n}})\ ;\ i=0(1)k, \qquad (2.21a)$$

where

$$\alpha_{5,1} = -1/5{,}040,\ \alpha_{5,2} = 1/21{,}600,\ \alpha_{5,3} = -1/190{,}080 \ . \qquad (2.21b)$$

$$\text{(b)} \quad y_i^{(2)} - S_i^{(2)} = \sum_{m=1}^{N_2} \beta_{5m,} h^{2+2m} u_i^{(4+2m)} + 0(h^{4+\hat{n}})\ ;\ i=0(1)k, \qquad (2.22a)$$

where

$$\beta_{5,1} = -1/720 \quad , \quad \beta_{5,2} = 1/3,360 \quad , \quad \beta_{5,3} = -1/86,400 \quad . \tag{2.22b}$$

$$\text{(c)} \quad y_i^{(3)} - S_i^{(3)} = \sum_{m=1}^{N_1} \gamma_{5,m} h^{2+2m} y_i^{(5+2m)} + 0(h^{3+\hat{n}}) \; ; \; i = 0(1)k \; , \tag{2.23a}$$

where

$$\Upsilon_{5,1} = 1/240 \quad , \Upsilon_{5,2} = -11/30,340 \; , \; \Upsilon_{5,3} = 1/28,800 \tag{2.23b}$$

Proof. The cubic spline result (2.20) is a trivial generalization of a result of [11,p.572], and is established by substituting the expressions for $S_i^{(2)}$, given by (2.19) with $r = 2$, into the cubic spline relations

$$S_0^{(1)} = -\frac{h}{3} S_0^{(2)} - \frac{h}{6} S_1^{(2)} + \frac{1}{h} (y_1 - y_0)$$

and

$$S_i^{(1)} = \frac{h}{6} S_{i-1}^{(2)} + \frac{h}{3} S_i^{(1)} + \frac{1}{h} (y_i - y_{i-1}) \; ; \quad i = 1(1)k$$

The results (2.21) - (2.23) are established in a simlar manner by substituting the expressions for $S_i^{(4)}$, given by (2.19) with r=3, into appropriate quintic spline identities. For example, the quintic spline relations

$$S_i^{(2)} = -\frac{h^2}{120} \left\{ S_{i-1}^{(4)} + 8S_i^{(4)} + S_{i+1}^{(4)} \right\} + \frac{1}{h^2} \delta^2 y_i \; ; \; i = 1(1)k - 1 \quad ,$$

in conjunction with (2.19), lead easily to the result (2.22), for i=1(1)k-1. (For the derivation of odd degree spline identities, see [1] and [7]. For a useful list of quintic spline relations, see [4].)

Lemma 2.3. Let S be either a cubic $(r = 2)$ or quintic $(r = 3)$ spline. If $y \in C^{2r+n}[a,b]$, $n \geq 1$, and the end conditons of S are of order $p \geq 2 + \hat{n}$, where $\hat{n} = \min\{n,4\}$ then, for $i = 0(1)k\text{-}1$,

$$y_i^{(2r-1)} - S^{(2r-1)}(x_i+) = \begin{cases} -\frac{h}{2}\, y_i^{(2r)} + 0(h^2)\,, & \text{when } n = 1\,, \\ -\frac{h}{2}\, y_i^{(2r)} - \frac{h^2}{12}\, y_i^{(2r+1)} + 0(h^{n+1})\,, & \text{when } n = 2,\ 3, \\ -\frac{h}{2}\, y_i^{(2r)} - \frac{h^2}{12}\, y_i^{(2r+1)} + \frac{(4-r)}{720}\, h\, y_i^{(2r+3)} + 0(h^5)\,, & \\ \qquad\qquad \text{when } n \geq 4\,. & (2.24) \end{cases}$$

<u>Proof.</u> . As in Lucas [12,Theor.2]. That is,

$$S^{(2r-1)}(x_i+) = \frac{1}{h}\left\{ S_{i+1}^{(2r-2)} - S_i^{(2r-2)} \right\}$$

and hence, from (2.19)

$$S^{(2r-1)}(x_i+1) = \frac{1}{h}\left\{ y_{i+1}^{(2r-2)} - y_i^{(2r-2)} \right\} - \sum_{m=1}^{N} \ell_{2r-1,\ m}\, h^{2m-1} \Big(y_{i+1}^{(2r-2+2m)} - y_i^{(2r-2+2m)} \Big) + 0(h^{1+\hat{n}})\ ;\ i = 0(1)k\text{-}1\ ,$$

where $N = \min\{\, [\,(n+1)/2\,]\,, 2\}$. The result then follows by expanding the derivatives $y_{i+1}^{(j)}$, by Taylor series about x_i.

Apart from the requirement concerning the end conditions of S, the expansions (2.19) - (2.23) are the special cases $r = 2,3$, of the asymptotic expansions of odd degree periodic splines, derived by Lucas [12,Theorems 1 and 2], Because of this, the three Theorems 2.1-2.3 given below emerge easily from the analysis of [13]. In fact, apart from the requirement for the end conditions of S, the three theorems are essentially the special cases $r = 2,3$, of Theorems 2 - 4 in [13] .

<u>Theorem 2.1</u>. Let S be either a cubic $(r = 2)$ or quintic $(r = 3)$ spline. If $y \in C^{2r+n}[a,b]$, $n \geq 1$, and the end conditions of S are of order $p \geq 2 + \hat{n}$, where $\hat{n} = \min\{n,4\}$, then for $0 \leq \mu \leq 1$ and $0 \leq j \leq 2r$,

$$y^{(j)}(x_i + \mu h) = S^{(j)}(x_i + \mu h) + \sum_{m=0}^{\hat{n}} \left\{ \frac{h^{2r-j+m}}{(2r+m)!}\, y_i^{(2r+m)}\, P_m^{(j)}(\mu) \right\} + 0(h^{2r-j+\hat{n}})\ ;$$
$$i = 0(1)k-1\ , \qquad (2.25)$$

where the polynomials P_m ; $m = 0(1)3$, are as follows:

(i) When $r = 2$, i.e. when S is a cubic spline,

$$\left.\begin{array}{ll} P_0(\mu) = \mu^4 - 2\mu^3 + \mu^2 , & P_1(\mu) = \mu^5 - \frac{5}{3}\mu^3 + \frac{2}{3}\mu \\ P_2(\mu) - \mu^6 - \mu^2 , & P_3(\mu) = \mu^7 - \frac{7}{3}\mu^3 + \frac{10}{3}\mu \end{array}\right\} \qquad (2.26a)$$

(ii) When $r = 3$, i.e. when S is a quintic spline,

$$\left.\begin{array}{ll} P_0(\mu) = \mu^6 - 3\mu^5 + \frac{5}{2}\mu^4 - \frac{1}{2}\mu^2 , & P_1(\mu) = \mu^7 - \frac{7}{2}\mu^5 + \frac{7}{2}\mu^3 - \mu , \\ P_2(\mu) - \mu^8 - 7\mu^4 + 6\mu^2 , & P_3(\mu) = \mu^9 - \frac{21}{5}\mu^5 - 22\mu^3 + \frac{84}{5}\mu . \end{array}\right\} \qquad (2.26b)$$

<u>Proof</u>. Exactly as in Lucas [13,Theorems 1,2]. That is, by Taylor series expansion about the point x_i,

$$y^{(j)}(x_i + \mu h) - S^{(j)}(x_i + \mu h) = \sum_{m=0}^{2r-1-j} \frac{(\mu h)^m}{m!} \left\{ y_i^{(j+m)} - S_i^{(j+m)} \right\}$$

$$+ \sum_{m=2r-j}^{2r+\hat{n}-j-1} \frac{(\mu h)^m}{m!} y_i^{(j+m)} + 0(h^{2r-j+\hat{n}}) ; \; j = 0(1)2r-1 , \qquad (2.27)$$

and

$$y^{(2r)}(x_i + \mu h) = \sum_{m=0}^{\hat{n}-1} \frac{(\mu h)^m}{m!} y_i^{(2r+m)} + 0(h^{\hat{n}}) \qquad (2.28)$$

For $j = 0(1)2r-1$, the result (2.25) is established by substituting into (2.27) the expansions for $y_i^{(j+m)} - S_i^{(j+m)}$; $j = 0(1)\,2r-1-j$, given by (2.19) - (2.24). For $j = 2r$, the result follows at once from (2.28), by observing that $S^{(2r)}(x_i + \mu h) = 0$ and that $P_m^{(2r)}(\mu) = (2r+m)!\, u^m/m\,!$

As in Theorem 2.1, let S be either a cubic ($r=2$) or quintic ($r = 3$) spline, and denote by $\tilde{y}_i^{(2r+j)}$; j - 0,1,2, the following approximations to $y_i^{(2r+j)}$; $j = 0,1,2$:

(i) $$y_i^{(2r)} = \frac{1}{h^2}\,\delta^2\, S_i^{(2r-2)}$$

$$= \frac{1}{h^2}\left\{S_{i-1}^{(2r-2)} - 2S_i^{(2r-2)} + S_{i+1}^{(2r-2)}\right\}; \qquad i = 1\,(1)\,k\text{-}1 \quad , \qquad (2.29a)$$

(ii) $$\tilde{y}_i^{(2r+1)} = \frac{1}{2h^3}\left\{\delta^2 S_{i+1}^{(2r-2)} - \delta^2 S_{i-1}^{(2r-2)}\right\}$$

$$= \frac{1}{2h^3}\left\{- S_{i-2}^{(2r-2)} + 2S_{i-1}^{(2r-2)} - 2S_{i+1}^{(2r-2)} + S_{i+2}^{(2r-2)}\right\};$$

$$i = 2\,(1)\,k\text{-}2 \quad , \qquad (2.29b)$$

(iii) $$\tilde{y}_i^{(2r+2)} = \frac{1}{h^4}\,\delta^4\, S_i^{(2r-2)}$$

$$= \frac{1}{h^4}\left\{- S_{i-2}^{(2r-2)} - 4S_{i-1}^{(2r-2)} + 6S_i^{(2r-2)} - 4S_{i+1}^{(2r-2)} + S_{i+2}^{(2r-2)}\right\};$$

$$i = 2\,(1)\,k-2 \qquad (2.29c)$$

Then, we have the following:

<u>Theorem 2.2.</u> With the notation introduced above, if $y \in C^{2r+n}[a,b]$, $n \geq 1$, and the end conditions of S are of order $p \geq 2 + \hat{n}$, where $n = \min\{n, 2r\}$, then:

(i) $$y_i^{(2r)} = \tilde{y}_i^{(2r)} + 0(h^{\hat{n}})\;;\; i = 1\,(1)\,k-1 \qquad (2.30)$$

(ii) For $n \geq 2$,

$$y_i^{(2r+1)} = \tilde{y}_i^{(2r+1)} + 0(h^{\ell})\;;\; i = 2\,(1)\,k-1 \quad , \qquad (2.31)$$

where $\ell = 1$ when $n = 2$, and $\ell = 2$ when $n > 2$.

(iii) For $n \geq 3$,

$$y_i^{(2r+2)} = \tilde{y}_i^{(2r+2)} + 0(h^{m})\;;\; i = 2\,(1)\,k-2 \quad , \qquad (2.32)$$

where $m = 1$ when $n = 3$, and $m = 2$ when $n > 3$.

Proof. See Lucas [12,Theor.3]. (The result (2.30) can be established directly by considering

$$E_{2r-1,\,i} = \frac{1}{h^2}\left\{ S_{i-1}^{(2r-2)} - 2S_i^{(2r-2)} + S_{i+1}^{(2r-2)} \right\} - y_i^{(2r)} \;;\; r = 2,3 \quad .$$

With $N = \min\{\,[\,(n+1)/2\,]\,,r\}$, this gives

$$E_{2r-1,\,i} = \frac{1}{h^2}\left\{ \lambda_{2r-1,i-1}^{\{N\}} - 2\lambda_{2r-1,i}^{\{N\}} + \lambda_{2r-1,i+1}^{\{N\}} \right\} + \varepsilon_{2r-1,\,i}^{\{N\}} \;;\; i = 1(1)\; k-1 \quad ,$$

where $\lambda_{2r-1,i}^{\{N\}} = 0(h^{2+\hat{n}})$ and where, by Taylor series expansion about x_i, $\varepsilon_{2r-1,\,i}^{\{N\}} = 0(h^{\hat{n}})$. This establishes (2.30). The results (2.31) and (2.32) then follow immediately from (2.30), because when $y \in C^{2r+n}[a,b]$, $n \geq 2$, then

$$y_i^{(2r+1)} = \frac{1}{h}\left\{ y_{i+1}^{(2r)} - y_{i-1}^{(2r)} \right\} + 0(h^{\ell}) \quad ,$$

and when $y \in C^{2r+n}[a,b]$, $n \geq 3$, then

$$y_i^{(2r+2)} = \frac{1}{h^2}\,\delta^2\, y_i^{(2r)} + 0(h^m).\,)$$

The result (2.30) is quite well-known; see e.g. [5], [8[, [6] and [11]. (The first reference of (2.30) is [5], where the cubic (r=2) result is derived for knots well away from the two end points of [a,b].)

Theorem 2.3. Let S be either a cubic (r=2) or quintic (r=3) spline, let $\tilde{y}_i^{(2r+m)}$; m = 0,1,2, denote the derivative approximations (2.29), and let P_m ; m = 0,1,2, be the polynomials (2.26). If $y \in C^{2r+n}[a,b]$, $n \geq 1$, and the end condiitions of S are of order $p \geq 2 + \hat{n}$, where $\hat{n} = \min\{n,3\}$, then for $0 \leq \mu \leq 1$ and $0 \leq j \leq 2r$.

$$y^{(j)}(x_i + \mu h) = S^{(j)}(x_i + \mu h) + \sum_{m=0}^{\hat{n}-1}\left\{ \frac{h^{2r-j+m}}{(2r+m)!}\, y_i^{(2r+m)}\, P_m^{(j)}(u) \right\} + 0(h^{2r-j+\hat{n}}) \quad ,(2.33)$$

where i ranges from 1 to k-1 when n = 1, and from 2 to k-2 when $n \geq 2$.

Proof. This is a direct consequence of Theorems 2.1 and 2.2.

Theorem 2.3 extends the main result of Lucas [13,Theor.4] to non-periodic cubic and quintic spline interpoxlation, but (2.33) does not hold over the whole of [a,b]. That is, (2.33) does not hold in $[x_0, x_1)$ when n= 1, and in $[x_0,x_i)$, $[x_1,x_2)$ and $(x_{k-1}, x_k]$ when $n\geq 2$. In order to extend the theorem over the whole of [a,b] we need to have appropriate estimates for $y_0^{(2r)}$ and $y_i^{(2r+1)}$, $y_i^{(2r+2)}$; $i = 0,1,k-1$. Such estimates can be derived, by elementary forward and backward difference techniques, from the approximations $\tilde{y}_i^{(2r)}$; i = 1(1)k-1, given by (2.29a). For example, under the hypotheses of Theorem 2,3, we have the following:

(i) If $n \geq 1$ then

$$y_0^{(2r)} = \tilde{y}_1^{(2r)} + 0(h) \tag{2.34}$$

(ii) If $n \geq 2$ then,

$$y_0^{(2r)} = 2\tilde{y}_1^{(2r)} - \tilde{y}_2^{(2r)} + 0(h^2), \quad \text{(see [6] and [7])}, \tag{2.35a}$$

$$y_0^{(2r+1)} = \frac{1}{h}(\tilde{y}_2^{(2r)} - \tilde{y}_1^{(2r)}) + 0(h), \tag{2.35b}$$

$$y_1^{(2r+1)} = \frac{1}{h}(\tilde{y}_2^{(2r)} - \tilde{y}_1^{(2r)}) + 0(h), \tag{2.35c}$$

and

$$y_{k-1}^{(2r+1)} = \frac{1}{h}(\tilde{y}_{k-1}^{(2r)} - \tilde{y}_{k-2}^{(2r)}) + 0(h). \tag{2.35d}$$

(iii) If $n \geq 3$ then,

$$y_0^{(2r)} = 3\tilde{y}_1^{(2r)} - 3\tilde{y}_2^{(2r)} + \tilde{y}_3^{(2r)} + 0(h^3), \tag{2.36a}$$

$$y_0^{(2r+1)} = \frac{1}{h}\left\{-\frac{5}{2}\tilde{y}_1^{(2r)} + 4\tilde{y}_2^{(2r)} - \frac{3}{2}\tilde{y}_3^{(2r)}\right\} + 0(h^2), \tag{2.36b}$$

$$y_1^{(2r+1)} = \frac{1}{2h}\left\{-3\tilde{y}_1^{(2r)} + 4\tilde{y}_2^{(2r)} - \tilde{y}_3^{(2r)}\right\} + 0(h^2), \tag{2.36c}$$

$$y_{k-1}^{(2r+1)} = \frac{1}{2h}\left\{3\tilde{y}_{k-1}^{(2r)} - 4\tilde{y}_{k-2}^{(2r)} + \tilde{y}_{k-3}^{(2r)}\right\} + 0(h^2), \tag{2.36d}$$

$$y_0^{(2r+2)} = \frac{1}{h^2}\left\{\tilde{y}_1^{(2r)} - 2\tilde{y}_2^{(2r)} + \tilde{y}_3^{(2r)}\right\} + 0(h), \tag{2.36e}$$

$$y_1^{(2r+2)} = \frac{1}{h^2}\left\{ \tilde{y}_1^{(2r)} - 2\tilde{y}_2^{(2r)} + \tilde{y}_3^{(2r)} \right\} + 0(h) , \qquad (2.36f)$$

and

$$y_{k-1}^{(2r+1)} = \frac{1}{h^2}\left\{ \tilde{y}_{k-1}^{(2r)} - 2\tilde{y}_{k-2}^{(2r)} + \tilde{y}_{k-3}^{(2r)} \right\} + 0(h) \qquad (2.36g)$$

The results (2.34) - (2.36) provide suitable estimates that can be used to replace the derivatives $y_0^{(2r)}$ and $y_i^{(2r+1)}$, $y_i^{(2r+2)}$; $i = 0,1,k-1$, in (2.25), and thus extend Theorem 2.3 to the whole of [a,b]. The main result of this section may be stated as follows.

<u>Theorem 2.4</u> With the notation of Theorem 2.3, Let $Y_M^{(j)}$; $1 \le M \le 3$, denote the corrected approximations

$$Y_M^{(j)}(x_i + \mu h) = S^{(j)}(x_i + \mu h) + \sum_{m=0}^{M-1} \left\{ \frac{h^{2r-j+m}}{(2r+m)!} \tilde{y}_i^{(2r+m)} P_m^{(j)}(\mu \right\} ;$$

$$i = 0(1)k-1 , \quad (2.37)$$

where the derivative estimates $\tilde{y}_0^{(2r)}$ and $\tilde{y}_i^{(2r+1)}$, $\tilde{y}_i^{(2r+2)}$; $i = 0, 1, k-1$, depend on the number M of correction terms used and are defined as follows:

(i) When M=1, $\tilde{y}_0^{(2r)}$ is the approximation defined by (2.34).

(ii) When $M = 2$, $\tilde{y}_0^{(2r)}$ and $y_i^{(2r+1)}$; $i = 0,1,k-1$ are the approximations defined by (2.35).

(iii) When $M = 3$, $\tilde{y}_0^{(2r)}$ and $y_i^{(2r+1)}$, $\tilde{y}_i^{(2r+2)}$; $i = 0,1,k-1$, are the approximations defined by (2.36).

Then, for $0 \le \mu \le 1$ and $0 \le j \le 2r$,

$$y^{(j)}(x_i + \mu h) = Y_M^{(j)}(x_i + \mu h) + 0(h^{2r-j+M}) \; ; i = 0(1)k-1 , \qquad (2.38)$$

provided that $y \in C^{2r+M}[a,b]$ and the end conditions of S are of order $p \ge 2 + M$.

We end this section by making the following general remarks.

<u>Remark 2.1</u> The result (2.38) should be compared with the result

$$y^{(j)}(x) = S^{(j)}(x) + 0(h^{2r-j}) \quad , \quad x \in [a,b] \; ; j = 0(1)2r-1 \qquad (2.39)$$

which holds when $y \in C^{2r}[a,b]$ and the end conditions of S are or order $p \geq 2$.

<u>Remark 2.2</u> The special case M= 1 of Theorem 2 .4, i.e. the case corresponding to the use of one correction term only, is established by different arguments to those of the present paper in [2] for cubic splines, and in [3] for quintics.

<u>Remark 2.3</u> In both the cubic $(r=2)$ and quintic $(r=3)$ cases, the even derivatives of the polynomials P., and P_3 in (2.25) are zero when $\mu = 0$, i.e.

$$P_1^{(j)}(0) = P_3^{(j)}(0) = 0 \; ; \; j = 0(2)\,2r$$

This implies the following:

(i) If $y \in C^{2r+2}[a,b]$ and the end conditions of S are of order $p \geq 4$ then for $j = 0(2)2r$,

$$y_i^{(j)} = Y_i^{(j)}(x_i) + 0(h^{2r-j+2}) \quad ; \quad i = 0(1)k-1 \qquad (2.40)$$

(ii) If $y \in C^{2r+4}[a.b]$ and the end conditions of S are of order $p \geq 6$ then, for $j = 0(2)\,2r$, $y_i^{(j)} = Y_3^{(j)}(x_i) + 0(h^{2r-j+4})$; $i = 0(1)k-1$, i.e.

$$y_i^{(j)} = S_i^{(j)} + \frac{h^{2r-j}}{(2r)!} P_0^{(j)}(0)\,\tilde{y}_i^{(2r)} + \frac{h^{2r-j+2}}{(2r+2)!} P_2^{(j)}(0)\,\tilde{y}_i^{(2r+2)} \; 0(h^{2r-j+4}) \; ;$$

$$i = 0(1)k-1 \qquad (2.41)$$

<u>Remark 2.4</u> The zeros of the polynomials $P_0^{(j)}$; $j = 1(1)2r-1$, give the points in $[x_i, x_{i+1}]$ where the derivatives $S^{(j)}$; $j = 1(1)2r-1$, of S display

"superconvergence". That is, if $y \in c^{2r+1}[a.b]$ and the end conditions of S are of order $p \geq 3$ then,

$$y^{(j)}(x_j + \mu_j h) = S^{(j)}(x_i + \mu_i h) + 0(h^{2r-j+1}) \; ; i = 1(1)k\text{-}1 \qquad (2.42)$$

where μ_i ; $j = 1(1)2r—1$, denote respectively the zeros of the polynomials $P_0^{(j)}$; $j = 1(1)2r\text{-}1$, in $[0,1]$. These zeros are as follows:

(i) If $r = 2$, i.e. if S is a cubic spline, then

$$\mu_1 = 0{,}1/2.1 \;, \quad \mu_2 = (3 \pm \sqrt{3}/6 \,, \quad \mu_3 = 1/2 \;. \qquad (2.43)$$

(ii) If $r = 3$, i.e. if S is a quintic spline, then

$$\left.\begin{array}{l} \mu_1 = 0\,,1/2\,,1\,, \quad \mu_2 \; 1 \pm \sqrt{1 - 4\sqrt{30}}\,, \quad \mu_3 = 0{,}1/2\,,1\,, \\ \\ \mu_4 = (3 \pm \sqrt{3}\,/6\,, \quad \mu_5 = 1/2\;. \end{array}\right\} \qquad (2.44)$$

Remark 2.5 Formula (2.37) with $\mu = 0$ gives improved approximations to the derivatives $y_i^{(j)}$; $j = 1(1)2r$, in terms of the values of the derivatives of S at the knots. For example, we have the following:

(i) If $y \in C^{2r+2}[a,b]$ and the end conditions of S are of order $p \geq 4$ then (2.40) gives,

$$y_i^{(2r-2)} = \frac{1}{12}\left\{S_{i-1}^{(2r-2)} + 10S_i^{(2r-2)} + S_{i+1}^{(2r-2)}\right\} + 0(h^4) \; ; i = 1(1)\,k-1 \;. \qquad (2.45)$$

(ii) If $y \in C^{2r+3}[a,b]$ and the end conditions of S are of order $p \geq 5$ then, with $M = 3$, (2.38) gives

$$y_i^{(2r-2)} = \frac{1}{24}\left\{S_{i-2}^{(2r-2)} - 14S_{i\text{-}1}^{(2r-2)} + 14\,S_{i+1}^{(2r-2)} - S_{i+2}^{(2r-2)}\right\} + 0(h) \;;$$

$$i = 2(1)k\text{-}2 \;.\square \qquad (2.46)$$

Remark 2.6 The requirement that the end conditions of S are of order $p \geq 2 + M$ is a necessary condition for the result (2.38) to hold over the whole of [a,b]. However, a sufficient condition for (2.38) to hold in any subinterval $[x_\ell , x_{\ell+1}]$, $0 \leq \ell$ k-1 , is that

$$\lambda^{\{N\}}_{2r-1,i} = 0(h^{2+M})$$

for i = and $i = \ell+1$, where $N = \min\{[(M+1)/2], r\}$. This is apparent from the analysis leading to Theorem 2.4.

<u>Remark 2.7</u> If $y \in C^{2r+M}[a,b]$, but the end conditions of S are of order $p < 2 + M$, then we expect (2.38) to hold only at subintervals "sufficiently" far from the two end points of [a,b]. In the cubic spline case, such subintervals can be determined easily by using the matrix technique of Kershaw [10]. The effect that "inferior" end conditions have on the quality of the corrected approximations (2.37) is considered in greater detail in Section 3.

3. <u>End Conditions</u>

If y is sufficiently smooth then Theorem 2.4 gives a method for the a posteriori correction of interpolating cubic and quintic splines, provided that the spline end conditions are chosen appropriately. More precisely, formula (2.38) gives improved approximations over the whole interval of interpolation, provided that the 2r-2 end conditions of the cubic (r=2) or quintic (r=3) spline S are of sufficiently high order in the sense of Definition 2.1.

In the cubic case, the end condition criterion (2.18), of Definition 2,1, is exactly the same as that used in an earlier paper by Lucas [11], in connection with the computation of improved cubic spline derivative approximations of the form (2.29a), (2.45) and (2.46). Also, the problem of constructing cubic spline end conditions of high order is studied fully in Papamichael and Worsey [14]. For example, the following cubic spline results emerge easily from the analysis of [14]; see also [11,p.577] and Remark 3.1 below.

(i) The end conditions

$$\Delta^{j+2} S_0^{(2)} = \Delta^{j+2} S_k^{(2)} = 0 \ , \quad 0 \le j \le 2 \ , \tag{3.1}$$

are of order $p = 2+j$

(ii) The end conditions

$$\left.\begin{aligned} &\Delta^{j} S_0^{(2)} = \Delta^{j} y_0^{(2)} \ , \\ &\Delta^{j} S_k^{(2)} = \Delta^{j} y_k^{(2)} \ , \ 0 \le j \le 2 \ , \end{aligned}\right\} \tag{3.2}$$

are of order $p = 2 + j$.

(iii) The end conditions

$$\left.\begin{aligned} &\Delta^{j} S_0^{(2)} = \Delta^{j} y_0^{(1)} \ , \\ &\Delta^{j} S_k^{(1)} = \Delta^{j} y_k^{(1)} \ , \ 0 \le j \le 2 \ , \end{aligned}\right\} \tag{3.3}$$

are of order $p = 3 + j$.

End conditions of higher order can be obtained by simply increasing the exponent j in (3.2) and (3.3). For example, with $j = 3$ the orders of (3.2) and (3.3) are respectively $p = 5$ and $p = 6$. Unfortunately, such end conditions require the values of $y^{(2)}$ or $y^{(1)}$ at an excessive number of knots, and it is unlikely that this information will be available in an interpolation problem. However, it is always possible to construct high order end conditions that require derivative information only at the two end points x_0 and x_k. For example, the following are established in [14].

Let

$$\underline{C}_1 = [1313,-2888,1866,-320,29] \quad ,$$

$$\underline{C}_2 = [-1187,-864,2376,-352,27] \quad ,$$

$$\underline{y}_0 = [y_0, y_1, y_2, y_3, y_4] \, ,$$

and

$$\underline{y}_k = [y_k, \; y_{k-1}, y_{k-2}, y_{k-3}, y_{k-4}]$$

Then, the end conditions

$$\left.\begin{aligned} 144S_0^{(2)} + 876S_1^{(2)} &= \frac{1}{h^2}\left\{\underline{C}_1\,\underline{y}_0^T - 60h^2\,y_0^{(2)}\right\} , \\ 876S_{k-1}^{(2)} + 144S_k^{(2)} &= \frac{1}{h^2}\left\{\underline{C}_1\,\underline{y}_k^T - 60h^2\,y_k^{(2)}\right\} , \end{aligned}\right\} \qquad (3.4)$$

are of order $p = 5$. Also, the end conditions

$$\left.\begin{aligned} S_0^{(2)} + 2S_1^{(2)} &= \frac{1}{846h^2}\left\{\underline{C}_2\,\underline{y}_0^T - 2940h\,y_0^{(1)} - 360h^2\,y_0^{(2)}\right\} , \\ 2S_{k-1}^{(2)} + S_k^{(2)} &= \frac{1}{846h^2}\left\{\underline{C}_2\,\underline{y}_k^T + 2940hy_k^{(1)} - 360h^2\,y_k^{(2)}\right\} , \end{aligned}\right\} \qquad (3.5)$$

are of order $p = 6$.

Remark 3.1 The cubic spline end conditions (3.1) - (3.5) are either of the form

$$\left.\begin{aligned} \sum_{i=2}^{2} \alpha_i\, S_i^{(2)} &= \frac{1}{h^2}\left\{\sum_{i=0}^{4} a_i y_i + h\sum_{i=0}^{2} b_i y_i^{(1)} + h^2 \sum_{i=0}^{2} c_i y_i^{(2)}\right\} , \\ \sum_{i=2}^{2} \alpha_i\, S_{k-1}^{(2)} &= \frac{1}{h^2}\left\{\sum_{i=0}^{4} a_i y_{k-1} - h\sum_{i=0}^{2} b_i y_{k-i} + h^2 \sum_{i=0}^{2} c_i y_{k-i}^{(2)}\right\} , \end{aligned}\right\} \qquad (3.6)$$

or else they can be easily expressed in this form, by using standard cubic spline identities. The order p of such end conditions can be determined as follows.

Let $y \in C^{8}[a,b]$. Then, with the notation (2.12), the end conditions (3.6) give

$$\sum_{i=0}^{2} \alpha_i \lambda_{3,i}^{\{2\}} = E_0 \text{ and } \sum_{i=0}^{2} \alpha_i \lambda_{3,k-1}^{\{2\}} = E_k \;, \tag{3.7}$$

where the order of E_0 and E_k can be determined easily by Taylor series expansion. Suppose that

$$E_i = 0\,(h^s) \quad ; \quad i = 0,k \,, \tag{3.8}$$

and observe that (3.7) together with the equations (2.13), where $N = 2$ and $\hat{n} = 4$, constitute a $(k-1) x\ (k-1)$ linear system for the parameters $\lambda_{3,i}^{\{2\}}$; $i = 0\,(1)k$. Let A be the matrix of coefficients of this Linear system. Then, clearly,

$$p. = \min\{s,4\} \quad ,$$

provided that A^{-1} is uniformly bounded. (See also the proofs of Theorems 3.1 and 3.2 below.) □

All the cubic spline results of [14] can be generalized in an obvious way to quintic splines, but the analysis is much more laborious in the qunitic case. However, the following quintic spline generalizations of the end conditions (3.1) - (3.3) can be verified easily, by using a technique similar to that described in Remark 3.1.

(i) The end cond it ions

$$\Delta^{2+j} S_i^{(4)} = \nabla^{2+j} S_{k-i}^{(4)} = 0 \,, \qquad 0 \le j \le 2 \,; \;\; i = 0,1 \,, \tag{3.9}$$

are of order $p = 2 + j$; see [4].

(ii) Let $\ell = 2$ or $\ell = 4$. Then, the end conditions

$$\left.\begin{aligned} \Delta^j S_0^{(\ell)} &= \Delta^j y^{(\ell)} \ , \\ \nabla^j S_{k-i}^{(\ell)} &= \nabla^j y_{k-i}^{(\ell)} \ , \ 0 \le j \le 2 \ , \end{aligned}\right\} \ i = 0, 1 \ , \tag{3.10}$$

are of order $p = 2 + j$.

(iii) Let $\ell=1$ or $\ell = 3$. Then, the end. conditions

$$\left.\begin{aligned} \Delta^j S_i^{(\ell)} &= \Delta^j y^{(\ell)} \\ \nabla^j S_{k-i}^{(\ell)} &= \nabla^j y_{k-i}^{(\ell)} \ , \ 0 \le j \le 2 \ , \end{aligned}\right\} \ i = 0, 1 \ , \tag{3.11}$$

are of order $p = 3 + j$.

In the remainder of this section we consider further the observation made in Remark 2.7, concerning the adverse effect that "inferior" end conditions have on the quality of the corrected approximations (2.37). We do this by taking S to be a cubic spline, and examining in detail the end conditions

$$S_i^{(2)} = y_i^{(2)} \ ; \ i = 0, k \ , \tag{3.12}$$

which are of order $p = 2$; see Eq. (3.2). First however, we recall the trivially obvious result that the end conditions (2.16) are of order $p=6$, and in the theorem below we express this result in a form suitable for comparison purposes.

<u>Theorem 3.1</u> Let $y \in C^{10}$ $[a,b]$ and let the end conditions of the cubic spline S be

$$S_i^{(2)} = y_i^{(2)} - \frac{h^2}{12} y_i^{(4)} + \frac{h^4}{360} y_i^{(6)} \ ; \ i = 0, k \ . \tag{3.13}$$

Then,

$$|\lambda_{3,i}^{\{2\}}| \le \frac{34}{8!} h^6 \| y^{(8)} \| + 0(h^8) \, ; \, i = 1(1)k-1 \, , \qquad (3.14)$$

where the $\lambda_{3,i}^{\{2\}}$ are given by (2.12), and $\| \cdot \|$ denotes the uniform function norm on [a,b].

Proof. The end conditions (3.13) imply that $\lambda_{3,0}^{\{2\}} = \lambda_{3,k}^{\{2\}} = 0$, Therefore, the equations (2.13) give the matrix system

$$A \underline{\lambda}_3^{\{2\}} = \underline{\varepsilon} \qquad (3.15a)$$

where A is the (k-1) x (k-1) tri-diagonal matrix

$$A = \begin{bmatrix} 4 & 1 & & \\ 1 & 4 & 1 & \\ & \cdot & \cdot & \cdot \\ & & 1 & 4 \end{bmatrix} , \qquad (3.15b)$$

$$\underline{\lambda}_3^{\{2\}} = \left[\lambda_{3,1}^{\{2\}} , \lambda_{3,2}^{\{2\}} , \ldots , \lambda_{3,k-1}^{\{2\}} \right]^T , \qquad (3.15c)$$

and

$$\underline{\varepsilon} = [\varepsilon_1 , \varepsilon_2 , \ldots , \varepsilon_{k-1}]^T , \qquad (3.15d)$$

with $\varepsilon_i = 0(h^6)$; $i = 1(1)k-1$. In fact, it can be shown easily that

$$\varepsilon_i = \frac{68}{8!} h^6 y_i^{(8)} + 0(h^8) \, ; \, i = 1\,(1)k-1 \, . \qquad (3.16)$$

The result (3.14) then follows at once from (31.5) and (3.16), because $\| A^{-1} \|_\infty \le 1/2$. □

Since (3.13) are the only end conditions for which $\lambda_{3,0}^{\{2\}} = \lambda_{3,k}^{\{2\}} = 0$, we may regard (3.16) as the "best" possible bound for the parameters $\lambda_{3,i}^{\{2\}}$. That is, for the purposes in the present paper, we may regard (3.13) as the end conditions that give the "best " corrected approximations.

Our main result concerning the inferior end conditions (3.12) is Theorem 3.2 below. This theorem is established by employing the matrix technique used by Kershaw [10], in connection with a study of the orders of approximation of the first derivatives of cubic splines at the knots.

Theorem 3.2 Let $y \in C^{10}[a,b]$ assume that $h< 1$ and let the end conditions of the cubic spline S be

$$S_i^{(2)} = y_i^{(2)} \; ; \; i = 0,k \tag{3.17}$$

Also, given two positive numbers σ and τ, where $0 < \tau \leq 1$, let

$$\lambda = -\{\sigma \log h + \log \tau\} \log(3+\sqrt{3}). \tag{3.18}$$

If $k > 2\lambda$ then

$$|\lambda_{3,1}^{\{2\}}| \leq \tau\left\{\frac{h^{2+\sigma}}{12}\|y^{(4)}\| + \frac{h^{4+\sigma}}{360}\|y^{(6)}\|\right\} + \frac{34}{8!}h^6\|y^{(8)}\| + 0(h^8)\;;$$

$$i = \ell(1)k-\ell \;, \tag{3.19}$$

where ℓ — $[1+\lambda]$ (As before, $[\cdot]$ denotes integer part.)

Proof. With the matrix notation introduced in the proof of Theorem 3.1, The linear system for the parameters $\lambda_{3,i}^{\{2\}}$ associated with the end condition (3.17), can be written as

$$A\underline{\lambda}_3^{\{2\}} = -\left\{\frac{h^2}{12}y_0^{(4)} - \frac{h^4}{360}y_0^{(6)}\right\}\underline{e}_1 - \left\{\frac{h^2}{12}y_{k-1}^{(4)} - \frac{h^4}{360}y_{k-1}^{(6)}\right\}\underline{e}_{k-1} + \underline{\varepsilon}\,, \tag{3.20}$$

where e. denotes the ith column of the (k-1) x (k-1) identity matrix. Let $(\underline{x})_i$ denote the ith compent of the vector $(\underline{x})$. Then (3.20) and (3.16) give

$$\left|\lambda_{3,i}^{\{2\}}\right| \leq \left\{\frac{h^2}{12}\|y^{(4)}\| + \frac{h^4}{360}y^{(6)}\|\right\}\left\{\left|(A^{-1}\underline{e}_1)_i\right| + \left|(A^{-1}\underline{e}_{k-1})_i\right|\right\}$$

$$+\frac{34}{8!}h^6\|y^{(8)}\| + 0(h^8)\;;\; i = 1(1)k-1 \;, \tag{3.21}$$

where, from the results of Kershaw [9],

$$|(A^{-1}\underline{e}_{-1})_i| < \alpha^{-i} \text{ and } |(A^{-1}\underline{e}_{k-1})_i| < \alpha^{-i-k} \; ; \; i=1(1)k-1\,, \qquad (3.22a)$$

with

$$\alpha = 2 + \sqrt{3}\,. \qquad (3,22b)$$

The result of the theorem follows from (3.21) and (3.22), because if ℓ - $[1 + \lambda]$ then

$$\alpha^{--i} \le \tau h^{\sigma} \;, \qquad \text{for } i \ge \ell \;,$$

and

$$\alpha^{i-k} \le \tau h^{\sigma} \;, \qquad \text{for } i \le k - \ell \;. \square$$

Let λ_M ; $M = 1,2,3$, denote the values of λ obtained from (3.18), by taking $\tau = 1$ and $\sigma = 1,2,3$, respectively. Also, let $\ell_M = [1 + \lambda_M]$; $M = 1,2,3$, and assume that k is sufficiently large so that $k > 2\ell_3$. Then, (3.19) implies that, for $0 \le \mu \le 1$ and $0 \le j \le 4$,

$$y^{(j)}(x_i + \mu h) = Y_M^{(j)}(x_i + \mu h) + 0(h^{4-j+M}) \; ; \; i = \ell_M(1)(k - \ell_M - 1)\,,\; M = 1,2,3, \qquad (3.23)$$

where $Y_M^{(j)}$ denote the corrected approximations (2.37), corresponding to the cubic spline S of Theorem 3.2; see Remark 2.6. In other words, if S satisfies the end conditions (3.17) then the result (2.38), of Theorem 2.4, holds for M= 1,2,3, in the sub intervals $[x_{\ell_M}, x_{k-\ell_M}]$; M= 1,2,3, respectively.

It is also of interest to determine the ranges of values of i for which the parameters $\lambda_{3,i}^{\{2\}}$ satisfy (3.19) with $\tau < 1$. For example, let $\hat{\ell} = [1 + \hat{\lambda}]\,\ell$ where $\hat{\lambda}$ is the value of λ obtained from (3.18) by taking $\sigma = 4$ and $\tau = (34x12) / 8!$. Then, for $k > 2\hat{\lambda}$, (3. 19) gives

$$|\lambda_{3,i}^{\{2\}}| \le \frac{34}{8i} h^6 \{\|y^{(4)}\| + \|y^{(6)}\|\} + 0(h^8) \; ; \qquad i = \hat{\ell}(1)k - \hat{\ell}\,. \qquad (3.24)$$

By comparison with (3.14), this result indicates that for $x \in [x_{\hat{\ell}}, x_{k-\hat{\ell}}]$ the corrected approximations $Y_3^{(j)}$ corresponding to the end conditions

(3.17) will be of comparable accuracy to those obtained by using the "best" end conditions (3.13).

For the purposes of illustration let $a=0$ and $b=1$, so that $h=1/k$. Then, (3.18) gives the following values of ℓ_1, ℓ_2, ℓ_3 and ℓ :

(i) When $k = 16$,

$$\ell_1 = 3, \quad \ell_2 = 5, \quad \ell_3 = 7 \tag{3.25}$$

(ii) When $k = 32$,

$$\ell_1 = 3, \quad \ell_2 = 6, \quad \ell_3 = 8, \quad \hat{\ell} = 15 \tag{3.26}$$

(iii) When $k = 64$,

$$\ell_1 = 4, \quad \ell_2 = 7, \quad \ell_3 = 10, \quad \ell = 17 \tag{3.27}$$

4. <u>Numerical Results</u>

In this section we present numerical results obtained by taking

$$y(x) = \exp(x) , \tag{4.1}$$

$$x_i = i/k \ ; \quad i = 0(1)k , \tag{4.2}$$

and computing corrected cubic and quintic spline approximations to y and its derivatives.

As before, we denote by $Y_M^{(j)}$, $1 \le M \le 3$, the corrected approx-.mations (2.37), and we also let $Y_0^{(j)} = S^{(j)}$. Then, the results given in Tables 4.1 -4.3 are estimates of the maximum errors

$$\max_{0\le i\le k-1} \left\{ \max_{x\in[x_i,x_{i+1}]} \left| y^{(j)}(x) - Y_M^{(j)}(x) \right| \right\} , \tag{4.3}$$

obtained by sampling $y^{(j)} - Y_M^{(j)}$ at 160 equally spaced points in [0,1]. We denote these estimates by $E_M^{\{j\}}(k)$ and, in the tables, we also list the computed values

$$R_M^{(j)}(k) = \log_2 \left\{ E_M^{(j)}(k) / E_M^{(j)}(2k) \right\}, \qquad (4.4)$$

giving the observed rates of convergence of the corrected approximations to $y^{(j)}$

Table 4.1 adn 4.2 contain the values $E_M^{\{j\}}$ (16) and $R_M^{\{j\}}$ (8) , obtained by using respectively a cubic and a quntic spline, each with end conditions of order p =5. More precisely, the results of these two tables correspond respectively to the use of the cubic end conditions (3.4), and the quintic end conditions (3,11) with j =2 and ℓ =1

The two Tables 4.3 and 4.4 both contain cubic spline results obtained by using the end conditions (3.17), which are of order p=2 only. Table 4.3 contains the values $E_M^{\{j\}}$ (16) and $R_M^{\{j\}}$ (8), which give respectively the maximum errors and the observed rates of convergence over the whole interval [0,1], In Table 4.4 we illustrate the conclusions that emerge from the result of Theorem 3.2 by listing the error estimates $\hat{E}_M^{\{j\}}$ (32), obtained by sampling $y^{(j)} - Y_M^{(j)}$ only at points of the subinterval [5/16,11/16]. We also illustrate the improved rates of convergence that the corrected approximations achieve away from the two end points, by listing the values

$$\hat{R}_M^{\{j\}}(32) = \log_2 \left\{ \hat{E}_M^{\{j\}}(32) / \hat{E}_M^{\{j\}}(64) \right\}, \qquad (4.5)$$

The numerical results confirm the theory, and illustrate the substantial improvements in accuracy that can be achieved by the method of a posteriori corrections. The results also show that improved approximations are obtained, over the whole interval of interpolation, only if the spline end conditions are of sufficiently high order.

All computations were carried out on an Eclipse MV/8000 computer, using programs written in Fortran with double precision working. Double length working on the Eclipse MV/8000 is between 16 and 17 significant figures.

TABLE 4. 1

Cubic spline S - End conditions of order $p = 5$.

	M= 0	M= 1	M= 2	M= 3
J = 0	1.05E-7 4.0	3.44E-9 4.9	8.85E-11 6.1	1.65E-11 6.9
J = 1	5.14E-6 3.0	2.17E-7 3.9	9.40E-9 5.2	9.32E-10 6.0
J = 2	8.31E- 4 1.9	2.99E-5 3.4	1.74E-6 4.2	4.84E-8 4.9
J = 3	8.06E-2 0.9	3.28E-3 2.1	1.07E-4 3.1	3.35E-6 4.3
J = 4	- -	1.48E-1 1 .0	8.41E-3 2.0	3.24E-4 3.1

Top entries : Values of $E_M^{\{j\}}$ *16* .

Bottom entries: Values of $R_M^{\{j\}}$ *(8) .*

Theoretical rate : $\underline{R}_M^{\{j\}} = 4 - J + M$.

TABLE 4.2

Quintic spline S — End conditions of order $p = 5$.

	M = 0	M = 1	M = 2	M = 3
J = 0	9.00E-12 5.8	3.49E-13 6.7	1.80E-14 7.8	1.33E-15* 8.4*
J = 1	4.40E-10 4.8	2.66E-11 5,7	1.25E-12 7.0	5.75E-14 7.7
J = 2	4.7 7E-8 3.8	3.21E-9 4.8	9.08E-11 5.8	2.94E-12 6.8
J = 3	4.59E-6 2.8	2.18E-7 3.8	3.88E-9 5.3	2.05E-10 5.8
J - 4	7.33E-4 1 .7	2.64E-5 3.2	7.9 7E-7 4. 1	1.08E-8 4.8
J = 5	7.12E-2 0.7	2.90E-3 2,0	6.61E-5 3.0	8.75E-7 3 .8
J = 6	- -	1.30E-1 0.9	3.6 2E-3 1 .9	3.7 0E-5 2.8

Top entries: Values of $E_M^{\{J\}}$ *16.*

Bottom entries: Values of $R_M^{\{J\}}$ *(8).*

Theoretical rate: $\underline{R}_M^{\{J\}} = 4 - J + M$.

(* *These entries are contaminated by rounding errors.*)

TABLE 4.3

Cubic spline. S - End conditions of order p = 2.

	M = 0	M = 1	M = 2	M = 3
J = 0	2.65E-7 4.0	1 .58E-7 4 . 0	1.50E-7 4 . 0	1.52E-7 4.0
J = 1	1 .46E-5 3. 1	1.03E-5 3.3	1.01E-5 3.4	1.04E-5 3.4
J = 2	1.07E-3 1 .9	7.9 7E-4 2. 1	9.09E-4 2.2	1.02E-3 2.2
J = 3	9.86E-2 0.9	2.38E-2 0.8	3.62E-2 1 . 1	4.6 7E-2 1 . 1

Top entries : Values of $E_M^{\{J\}}$ *16 .*

Bottom entries: Values of $R_M^{\{J\}}$ *(8) .*

Theoretical rate : $\underline{R}_M^{\{J\}}$ *= 4 - J + M . = 0, 1, 2, 3.*

TABLE 4.4

Cubic spline S — End conditions of order $p = 2$.

	M = 0	M = 1	M = 2	M = 3
J = 0	4.86E-9 4.0	8. 1 1E-1 1 5.0	9.59E-13 5.9	3.13E-14 5.6*
J = 1	4.78E-7 3.0	1.02E-8 4.0	2.65E-10 5.3	3.10E-12 5.8
T = 2	1.57E-4 2.0	4.36E-6 3.2	6.36E-8 4.1	3.54E-10 5.2
J = 3	3.03E-2 1 .0	7.4 7E-4 2. 1	9.03E-6 3.1	4.66E-8 4.0
J = 4		4.96E-2 1 .0	8.95E-4 2.1	3.8 7E-6 3.0

Top entries: Values of $\hat{E}_M^{\{J\}}$ *(32).*

Bottom entries: Values of $\hat{R}_M^{\{J\}}$ *(32).*

Theoretical rate: $\underline{\hat{R}}_M^{\{J\}} = 4 - J + M,$

(* *These entries are contaminated by rounding errors.*)

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